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Classical and quantum three-dimensional integrable systems with axial symmetry

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Abstract

We study the most general form of a three-dimensional classical integrable system with axial symmetry and invariant under the axis reflection. We assume that the three constants of motion are the Hamiltonian, H, with the standard form of a kinetic part plus a potential depending on the position only, the z-component of the angular momentum, L, and a Hamiltonian-like constant, $\widetilde{H}$, for which the kinetic part is quadratic in the momenta. We find the explicit form of these potentials compatible with complete integrability. The classical equations of motion, written in terms of two arbitrary potential functions, are separated in oblate spheroidal coordinates. The quantization of such systems leads to a set of two differential equations that can be presented in the form of spheroidal wave equations.

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1. Motivation and introduction

This is a paper on integrability of three-dimensional systems, both classical and quantum. Apart from being interesting by itself, this study can be extremely interesting for a field, which is rather far away from the present subject, the theory of the continuous media: gas, fluid and plasma. Those are mechanical systems with an infinite number of degrees of freedom and are certainly much more difficult than the mechanics of a particle in three dimensions. However, in the theory of continuous media, we can pose a problem of the existence of solitons: the steady solution with time-independent field with density $\rho(x)$ and velocities $\mathbf{v}(x)$. It could exist in the non-dissipative case only, as we neglect the viscosity. In this latter case, the particles which constitute the media should move in a self-consistent potential along its trajectories, which should be closed. Although this potential is the result of interparticle interaction, each single particle moves in an effective potential that provides the closed trajectory. From the analysis of the equations of motion in continuous media [1], it follows that the simplest shape of the solution is a toroid. Thus, we have to look for a three-dimensional single-particle integrable system including a toroidal shape. One of the goals of the present paper is to make a first step in this direction, as we describe all completely integrable systems with axial symmetry.

A complete integrability of systems with axial symmetry is achieved if we found two independent Hamiltonians, which commute with the component of the angular momentum along the symmetry axis and also among themselves. Our Hamiltonians have a kinetic part which is quadratic on the components of positions and momenta, plus a potential depending solely on the position.

First, we study this situation classically, where the commutation rule for observables is given by the vanishing of the Poisson bracket for these observables. Our first goal is to obtain the most general form of the potentials for the Hamiltonians under the just described conditions. In section 2, we arrive at the conclusion that these potentials should have very simple general respective forms in terms of the oblate spheroidal coordinates and two undetermined functions, one of each coordinate.

In terms of these coordinates, complete integration is now possible. We can show that the Hamilton–Jacobi characteristic function is now separable, i.e., it can be written as a sum of three functions each one in one variable. The expression for the characteristic function permits us to write the equations of motion on a form that depends on the choice of the undetermined functions. We describe these procedures in section 3. In this sense, we have found a general solution for this problem as treated classically.

We can also treat it from the point of view of quantum mechanics. We transform the classical Hamiltonian into quantum operators by making use of canonical quantization. Since the kinetic parts of the Hamiltonians are quadratic in positions and momenta, they give rise to self-adjoint operators and then, self-adjointness of these Hamiltonians can be studied for particular cases by using Kato–Rellich-type theorems [12]. In our analysis, we have studied the Schrödinger equations that correspond to both Hamiltonians for our particular choice of the undetermined functions. These Schrödinger equations become spheroidal wave equations, under a convenient change of variables. Spheroidal wave equations have been studied in the classical literature [5]. In section 5, we discuss their solutions.

Concrete examples and models of physical interest of the applicability of this method are under study.

2. The axial coordinates

Let us consider a three-dimensional classical system with the canonical coordinates $(p_i, x_i), i = 1, 2, 3$, where the commutators are defined by the usual Poisson brackets $\{A, B\} = \frac{\partial A}{\partial p_k}\frac{\partial B}{\partial x_k} - \frac{\partial A}{\partial x_k}\frac{\partial B}{\partial p_k}$, the sum being understood over repeated indices. The evolution of the system is described by the Hamiltonian

$$H = \frac{\mathbf{p}^2}{2m} + U(\mathbf{x}), \qquad \mathbf{x} = (x_1, x_2, x_3), \qquad \mathbf{p} = (p_1, p_2, p_3), \tag{2.1}$$

where $U(\mathbf{x})$ is a potential term. Let us assume the existence of two additional integrals of motion for this system, all of them in involution. The first one will be chosen as the angular momentum along the direction fixed by the unit vector $\mathbf{n} = (n_1, n_2, n_3)$:

$$L = \mathbf{n} \cdot (\mathbf{x} \times \mathbf{p}). \tag{2.2}$$

As for the second one we require to be quadratic in momentum and commuting with both, H and L. Then, it is straightforward to show that, provided we include also the reflection with respect to the **n**-axis as a discrete symmetry of our system, it must take the general expression for which the method for its derivation will be explained below:

$$\widetilde{H} = \frac{1}{2m} p_i g^{ik}(\mathbf{x}) p_k + \Phi(\mathbf{x}), \tag{2.3}$$

where the quadratic term has the 'metric'

$$g^{ik}(\mathbf{x}) = \delta^{ik}(\mathbf{x}\cdot\mathbf{n})^2 - \mathbf{x}\cdot\mathbf{n}(x_i n_k + x_k n_i) + (\mathbf{x}^2 - a^2) n_i n_k, \tag{2.4}$$

with a^2 being a real constant that we take positive. The quadratic term in the momenta can be written as

$$\frac{1}{2m}\left(L_\perp^2 - a^2 {p_\mathbf{n}}^2\right), \tag{2.5}$$

where $L_\perp$ is the perpendicular component of $\mathbf{L}$ to the **n**-axis, and $p_\mathbf{n} = \mathbf{p}\cdot\mathbf{n}$. Here we must point out that the operators (2.2) and (2.5) determine the prolate–oblate spheroidal coordinates that separate the Laplacian operator [9].

In order to obtain (2.3) and (2.4), we make the following considerations: first of all, we assume that the kinetic term in (2.3) given by $T := (p_i g^{ik}(\mathbf{x}) p_k)/(2m)$ is quadratic in p_k, which are the components of the momentum. Then, let us first assume that the symmetry axis is the z axis. A constant of motion is the L_3, i.e., the component of the angular momentum in the direction of this axis. The quadratic terms on x_i and p_k that commute with L_3 are

$$\left\{L_1^2 + L_2^2, L_3\right\} = 0, \qquad \left\{p_1^2 + p_2^2, L_3\right\} = 0, \qquad \left\{p_3^2, L_3\right\} = 0$$

and

$$\{L_1 p_1 + L_2 p_2, L_3\} = 0.$$

Invariance with respect to reflections with respect to the origin in the direction of the z axis excludes this last term, $L_1 p_1 + L_2 p_2$, so that the kinetic term T should be a linear combination of $L_1^2 + L_2^2$ (or equivalently $\mathbf{L}^2 - L_3^2$), $p_1^2 + p_2^2$ and p_3^2 (or equivalently, $\mathbf{p}^2$ and p_3^2). If we exclude the trivial free particle contribution $\mathbf{p}^2$, we conclude that T should be a linear combination of $\mathbf{L}^2 - L_3^2$ and p_3^2.

Finally, if we replace this symmetry axis by an arbitrary axis in the direction of the unit vector $\mathbf{n} := (n_x, n_y, n_z)$, we arrive at the same result provided that we replace L_3 by $\mathbf{L}\cdot\mathbf{n}$ and p_3 by $\mathbf{p}\cdot\mathbf{n}$. Then, after a cumbersome but straightforward calculation, we obtain T in the desired form. Note that the constant a^2 in (2.4) comes from the linear combination. Also, the quadratic dependence on $\mathbf{x}$ in the kinetic term in (2.3) comes solely from the contribution of the angular momentum to this quadratic term. Note that this quadratic dependence on the position coordinates comes after calculation and it is not a working hypothesis.

The commutation of L with H and $\widetilde{H}$ restricts the form of their potential terms

$$U(\mathbf{x}) = U(\mathbf{x}^2, (\mathbf{x}\cdot\mathbf{n})^2), \qquad \Phi(\mathbf{x}) = \Phi(\mathbf{x}^2, (\mathbf{x}\cdot\mathbf{n})^2) \tag{2.6}$$

while the commutation of H and $\widetilde{H}$ leads to the equations

$$\partial_i \Phi(\mathbf{x}) = g^{ik}(\mathbf{x}) \partial_k U(\mathbf{x}). \tag{2.7}$$

In order to deal with (2.7) we diagonalize the matrix $g^{ik}(\mathbf{x})$. Its eigenvalues $\lambda(\mathbf{x})$ and eigenvectors $A(\mathbf{x})$ are obtained from the matrix equation

$$(g^{ik}(\mathbf{x}) - \lambda(\mathbf{x})\delta^{ik}) A_k(\mathbf{x}) = 0. \tag{2.8}$$

We get the solutions

$$\lambda_\pm(\mathbf{x}) = \frac{\mathbf{x}^2 - a^2}{2} \pm \sqrt{\left(\frac{\mathbf{x}^2 - a^2}{2}\right)^2 + a^2(\mathbf{x}\cdot\mathbf{n})^2}, \qquad \lambda_0(\mathbf{x}) = (\mathbf{x}\cdot\mathbf{n})^2. \tag{2.9}$$

The corresponding eigenvectors can be expressed in the following way:

$$A_i^+(\mathbf{x}) = \partial_i \lambda_-(\mathbf{x}), \qquad A_i^-(\mathbf{x}) = \partial_i \lambda_+(\mathbf{x}), \qquad A_i^0(\mathbf{x}) = \frac{\mathbf{n}\times\mathbf{x}}{\mathbf{x}^2 - (\mathbf{x}\cdot\mathbf{n})^2} = \partial_i \varphi(\mathbf{x}), \tag{2.10}$$

where φ is the azimutal angle around the $\mathbf{n}$-axis. Let us write here some useful identities of these eigenvalue functions for future calculations,

$$\begin{aligned}(\partial_i\lambda_+)^2 &= \frac{4\lambda_+}{\lambda_+ - \lambda_-}(\lambda_+ + a^2)\\ (\partial_i\lambda_-)^2 &= -\frac{4\lambda_-}{\lambda_+ - \lambda_-}(\lambda_- + a^2)\\ (\partial_i\lambda_0)^2 &= -\frac{4}{a^2}\lambda_+\lambda_-.\end{aligned} \tag{2.11}$$

Since the eigenvectors (2.10) are orthogonal, it is natural to adopt as new orthogonal coordinates the set $\{\lambda_+(\mathbf{x}), \lambda_-(\mathbf{x}), \varphi(\mathbf{x})\}$, which are essentially the *oblate spheroidal* coordinates [9]. We turn to equation (2.7) for the potentials, now expressed in the new coordinate system. Taking into account that due to the geometric symmetry, L (2.2), U and Φ do not depend on φ, (2.7) becomes

$$\partial_i\lambda_+\partial_+\Phi + \partial_i\lambda_-\partial_-\Phi = g^{ik}\left(\partial_k\lambda_+\partial_+U + \partial_k\lambda_-\partial_-U\right), \tag{2.12}$$

where $\partial_\pm$ stand for $\frac{\partial}{\partial\lambda_\pm}$. Now, as g^{ik} is diagonal in the new coordinate basis $\{\nabla\lambda_-, \nabla\lambda_+, \nabla\varphi\}$, this equation decouples in

$$\partial_+\Phi = \lambda_-\partial_+U, \qquad \partial_-\Phi = \lambda_+\partial_-U \tag{2.13}$$

or

$$\partial_+(\Phi - \lambda_-U) = 0, \qquad \partial_-(\Phi - \lambda_+U) = 0. \tag{2.14}$$

This means that

$$\Phi - \lambda_+U = -f(\lambda_+), \qquad \Phi - \lambda_-U = -g(\lambda_-), \tag{2.15}$$

where $f(\lambda_+)$ and $g(\lambda_-)$ are arbitrary functions. The minus sign in the front of (2.15) intends that the next equations, derived from (2.15), can be written in a symmetric form with respect to the variables λ_+ and λ_-. These equations give an expression for the potentials as follows:

$$U = \frac{f(\lambda_+) - g(\lambda_-)}{\lambda_+ - \lambda_-}, \qquad \Phi = \frac{\lambda_-}{\lambda_+ - \lambda_-}f(\lambda_+) - \frac{\lambda_+}{\lambda_+ - \lambda_-}g(\lambda_-). \tag{2.16}$$

These are the most general expressions for the potentials $U(\mathbf{x})$ and $\Phi(\mathbf{x})$ compatible with $\{H, \widetilde{H}\} = 0$.

3. Separation of variables

3.1. The momentum as a gradient

Now we will recall here a property for a classical system in three dimensions having three integrals of motion (including the Hamiltonian) $h_i, i = 1, 2, 3$, in involution, thus being integrable:

$$h_i(\mathbf{p}, \mathbf{x}) = e_i, \qquad i = 1, 2, 3 \tag{3.1}$$

$$\{h_i, h_j\} \equiv \frac{\partial h_i}{\partial p_k}\frac{\partial h_j}{\partial x_k} - \frac{\partial h_i}{\partial x_k}\frac{\partial h_j}{\partial p_k} = 0, \qquad \forall\, i, j. \tag{3.2}$$

If we assume that the determinant of the matrix $\left(\frac{\partial h_i}{\partial p_k}\right)$ is nonvanishing, the inverse function theorem gives from (3.1) the expression of the momenta as functions of coordinates, at least locally:

$$p_i = f_i(\mathbf{x}, \mathbf{e}) \tag{3.3}$$

so that

$$h_i(\mathbf{f}(\mathbf{x}, \mathbf{e}), \mathbf{x}) = e_i. \tag{3.4}$$

Taking partial derivatives of this equation, we obtain

$$\frac{\partial h_i}{\partial p_k}\frac{\partial f_k}{\partial x_l} + \frac{\partial h_i}{\partial x_l} = 0 \quad \Longrightarrow \tag{3.5}$$

$$\frac{\partial h_i}{\partial p_k}\frac{\partial f_k}{\partial x_l}\frac{\partial h_m}{\partial p_l} = -\frac{\partial h_i}{\partial x_l}\frac{\partial h_m}{\partial p_l} = -\frac{\partial h_m}{\partial x_l}\frac{\partial h_i}{\partial p_l} \quad \Longrightarrow \tag{3.6}$$

$$\frac{\partial h_i}{\partial p_k}\frac{\partial f_k}{\partial x_l}\frac{\partial h_m}{\partial p_l} + \frac{\partial h_i}{\partial p_k}\frac{\partial h_m}{\partial x_k} = 0. \tag{3.7}$$

Note that the second identity in (3.6) comes after condition (3.2). According to our hypothesis, the determinant of the matrix $\left(\frac{\partial h_i}{\partial p_k}\right)$ is nonvanishing, so that the last relation can be simplified:

$$\frac{\partial f_k}{\partial x_l}\frac{\partial h_m}{\partial p_l} + \frac{\partial h_m}{\partial x_k} = 0. \tag{3.8}$$

From (3.5) and (3.8) we have

$$\frac{\partial h_m}{\partial p_l}\left(\frac{\partial f_k}{\partial x_l} - \frac{\partial f_l}{\partial x_k}\right) = 0 \tag{3.9}$$

hence

$$\frac{\partial f_k}{\partial x_l} = \frac{\partial f_l}{\partial x_k}. \tag{3.10}$$

This means that the vector field, in the variable $\mathbf{x}$, $\mathbf{f}(\mathbf{x}, \mathbf{e})$, has vanishing rotational, i.e.,

$$\mathbf{rot}_{\mathbf{x}}\mathbf{f}(\mathbf{x}, \mathbf{e}) = \mathbf{0}. \tag{3.11}$$

Therefore, if $\mathbf{f}(\mathbf{x}, \mathbf{e})$ is sufficiently regular, there exists a function $F(\mathbf{x}, \mathbf{e})$, locally defined, such that

$$f_l = \frac{\partial F(\mathbf{x}, \mathbf{e})}{\partial x_l} \qquad \text{i.e.} \quad \mathbf{grad}_{\mathbf{x}} F(\mathbf{x}, \mathbf{e}) = \mathbf{f}(\mathbf{x}, \mathbf{e}). \tag{3.12}$$

In conclusion, we have shown that if there are three integrals of motion (3.1) in involution, the momentum will be the gradient of the function F:

$$p_l = \frac{\partial F(\mathbf{x}, \mathbf{e})}{\partial x_l}. \tag{3.13}$$

These could be considered as a first class constraints. The function F is the characteristic function in the Hamilton–Jacobi approach. Note that this is a general property, valid for any dimension n.

3.2. The separation of F

We can apply the above results to our system by making the identification $h_1 = H, h_2 = L, h_3 = \widetilde{H}$. The next step is to show the separability of the function F in the variables $\{\lambda_\pm, \varphi\}$ introduced in the previous section. Then, if we apply the chain rule to (3.12) and make use of (2.10), we have

$$\mathbf{p} = \nabla\lambda_+\partial_+F(\lambda_+,\lambda_-,\varphi) + \nabla\lambda_-\partial_-F(\lambda_+,\lambda_-,\varphi) + \frac{\mathbf{n}\times\mathbf{x}}{\mathbf{x}^2-(\mathbf{x}\cdot\mathbf{n})^2}\partial_\varphi F(\lambda_+,\lambda_-,\varphi). \tag{3.14}$$

Since the vector fields in (2.10) are mutually orthogonal, multiplying both sides of the above expression by $\mathbf{n}\times\mathbf{x}$, we obtain

$$(\mathbf{n}\times\mathbf{x})\cdot\mathbf{p} = \mathbf{n}\cdot\mathbf{L} = \ell = \partial_\varphi F(\lambda_+,\lambda_-,\varphi), \tag{3.15}$$

where ℓ is the value of the integral of motion L corresponding to the angular momentum around $\mathbf{n}$. We conclude that

$$\mathbf{p} = \nabla\lambda_+\partial_+F(\lambda_+,\lambda_-,\varphi) + \nabla\lambda_-\partial_-F(\lambda_+,\lambda_-,\varphi) + \frac{\mathbf{n}\times\mathbf{x}}{\mathbf{x}^2-(\mathbf{x}\cdot\mathbf{n})^2}\ell \tag{3.16}$$

and therefore taking the square modulus in (3.16), we have that

$$\mathbf{p}^2 = (\nabla\lambda_+)^2(\partial_+F)^2 + (\nabla\lambda_-)^2(\partial_-F)^2 + \frac{\ell^2}{\mathbf{x}^2-(\mathbf{x}\cdot\mathbf{n})^2} = 2m\,(E-U)\,, \tag{3.17}$$

where E is the value of the constant of motion H. We recall that the expression (3.16) for the momentum vector field has been obtained in the basis that diagonalizes the metric matrix of components $\{g^{ik}\}$. In this basis and from (3.16), we straightforwardly compute

$$p_i g^{ik} p_k = \lambda_-(\nabla\lambda_+)^2(\partial_+F)^2 + \lambda_+(\nabla\lambda_-)^2(\partial_-F)^2 + \frac{(\mathbf{x}\cdot\mathbf{n})^2\ell^2}{\mathbf{x}^2-(\mathbf{x}\cdot\mathbf{n})^2} = 2m(\widetilde{E}-\Phi), \tag{3.18}$$

where $\widetilde{E}$ is the value of $\widetilde{H}$.

Next, we multiply equation (3.17) by λ_+, (3.18) by -1 and sum. Next we make the same procedure with λ_- instead. Finally, we get the following two expressions:

$$\begin{cases} (\nabla\lambda_+)^2(\lambda_+-\lambda_-)(\partial_+F)^2 = 2m(\lambda_+(E-U)-(\widetilde{E}-\Phi)) - \dfrac{\ell^2}{\mathbf{x}^2-(\mathbf{x}\cdot\mathbf{n})^2}(\lambda_+-(\mathbf{x}\cdot\mathbf{n})^2) \\ -(\nabla\lambda_-)^2(\lambda_+-\lambda_-)(\partial_-F)^2 = 2m(\lambda_-(E-U)-(\widetilde{E}-\Phi)) - \dfrac{\ell^2}{\mathbf{x}^2-(\mathbf{x}\cdot\mathbf{n})^2}(\lambda_--(\mathbf{x}\cdot\mathbf{n})^2). \end{cases} \tag{3.19}$$

Taking into account the definition (2.9) of $\lambda_\pm$, the next formulae are easily obtained:

$$\begin{aligned} &\mathbf{x}^2-(\mathbf{x}\cdot\mathbf{n})^2 = \lambda_+ + \lambda_- + a^2 + \frac{\lambda_+\lambda_-}{a^2} = \frac{1}{a^2}(\lambda_+ + a^2)(\lambda_- + a^2) \\ &\lambda_+ - (\mathbf{x}\cdot\mathbf{n})^2 = \lambda_+ + \frac{\lambda_+\lambda_-}{a^2} = \frac{\lambda_+}{a^2}(\lambda_- + a^2) \\ &\lambda_- - (\mathbf{x}\cdot\mathbf{n})^2 = \lambda_- + \frac{\lambda_+\lambda_-}{a^2} = \frac{\lambda_-}{a^2}(\lambda_+ + a^2) \\ &(\lambda_+-\lambda_-)(\nabla\lambda_+)^2 = 4\lambda_+(\lambda_+ + a^2) \\ &(\lambda_+-\lambda_-)(\nabla\lambda_-)^2 = -4\lambda_-(\lambda_- + a^2). \end{aligned} \tag{3.20}$$

Then, we carry (3.20) into (3.19) to get

$$\begin{aligned} &4\lambda_+(\lambda_+ + a^2)(\partial_+F)^2 = 2m(\lambda_+(E-U)-(\widetilde{E}-\Phi)) - \frac{\lambda_+\ell^2}{\lambda_+ + a^2} \\ &4\lambda_-(\lambda_- + a^2)(\partial_-F)^2 = 2m(\lambda_-(E-U)-(\widetilde{E}-\Phi)) - \frac{\lambda_-\ell^2}{\lambda_- + a^2}. \end{aligned} \tag{3.21}$$

We finally have

$$
\begin{aligned}
(\partial_+ F)^2 &= \frac{1}{4\lambda_+(\lambda_+ + a^2)}\left\{2m(\lambda_+(E-U) - (\widetilde{E}-\Phi)) - \frac{\lambda_+\ell^2}{\lambda_+ + a^2}\right\} \\
(\partial_- F)^2 &= \frac{1}{4\lambda_-(\lambda_- + a^2)}\left\{2m(\lambda_-(E-U) - (\widetilde{E}-\Phi)) - \frac{\lambda_-\ell^2}{\lambda_- + a^2}\right\}.
\end{aligned} \tag{3.22}
$$

Now, let us observe that in the right-hand side of (3.22) the combination $\lambda_+(E-U)-(\widetilde{E}-\Phi)$ cannot depend on λ_-. At the same time, $\lambda_-(E-U)-(\widetilde{E}-\Phi)$ will not depend on λ_+. Note that a similar behavior have arosen in (2.14). Hence, since $\partial_+ F$ and $\partial_- F$ depends only on λ_+ and λ_-, respectively, while the dependence on φ was given by (3.15), we conclude that we have managed to obtain the separation of F in the variables $\lambda_\pm$, φ, i.e.,

$$
F(\lambda_+, \lambda_-, \varphi) = A(\lambda_+) + B(\lambda_-) + \ell\varphi. \tag{3.23}
$$

Later, we shall find an explicit form for F in terms of a different set of variables.

3.3. The integration of the equations of motion

Our next objective is the integration of the following equations of motion:

$$
p_i = \partial_i F(\mathbf{x}). \tag{3.24}
$$

In other words, our goal is finding the explicit dependence of the variables $\lambda_\pm$ and φ with time, i.e., the functions $\lambda_\pm = \lambda_\pm(t)$ and $\varphi = \varphi(t)$. From (3.22) and (3.23), it is clear that

$$
(\partial_+ A(\lambda_+))^2 = P(\mathbf{x}, \lambda_+) \qquad (\partial_- B(\lambda_-))^2 = P(\mathbf{x}, \lambda_-), \tag{3.25}
$$

where we are using the shorthand notation

$$
P(\mathbf{x}, \lambda_\pm) = \frac{1}{4\lambda_\pm(\lambda_\pm + a^2)}\left\{2m(\lambda_\pm(E-U(\mathbf{x})) - (\widetilde{E}-\Phi(\mathbf{x}))) - \frac{\ell^2\lambda_\pm}{\lambda_\pm + a^2}\right\}. \tag{3.26}
$$

The chain rule along $\mathbf{p} = m\dot{\mathbf{x}}$ and (3.24) gives us

$$
\begin{aligned}
\dot{\lambda}_+ &= \partial_i\lambda_+\dot{x}_i = \frac{1}{m}\partial_i\lambda_+ p_i = \frac{1}{m}\partial_i\lambda_+\partial_i F \\
\dot{\lambda}_- &= \partial_i\lambda_-\dot{x}_i = \frac{1}{m}\partial_i\lambda_- p_i = \frac{1}{m}\partial_i\lambda_-\partial_i F \\
\dot{\varphi} &= \partial_i\varphi\dot{x}_i = \frac{1}{\mathbf{x}^2 - (\mathbf{x}\cdot\mathbf{n})^2}\frac{\ell}{m} = \frac{1}{m}\partial_i\varphi\partial_i F,
\end{aligned} \tag{3.27}
$$

where the dot denotes time derivative, as usual. Therefore,

$$
\begin{cases}
\dot{\lambda}_+ = \dfrac{1}{m}\partial_i\lambda_+\partial_i F = \dfrac{1}{m}(\partial_i\lambda_+)^2\partial_+ A(\lambda_+) = \dfrac{1}{m}\dfrac{4\lambda_+}{\lambda_+ - \lambda_-}(\lambda_+ + a^2)\partial_+ A(\lambda_+) \\
\dot{\lambda}_- = -\dfrac{1}{m}\dfrac{4\lambda_-}{\lambda_+ - \lambda_-}(\lambda_- + a^2)\partial_- B(\lambda_-) \\
\dot{\varphi} = \dfrac{a^2\ell}{m}\dfrac{1}{(\lambda_+ + a^2)(\lambda_- + a^2)} = \dfrac{a^2\ell}{m}\dfrac{1}{\lambda_+ - \lambda_-}\left(\dfrac{1}{\lambda_- + a^2} - \dfrac{1}{\lambda_+ + a^2}\right).
\end{cases} \tag{3.28}
$$

From the first two equations in (3.28),

$$
\begin{cases}
\dfrac{\dot{\lambda}_+}{\lambda_+\partial_+ A(\lambda_+)} = \dfrac{1}{m}\dfrac{4(\lambda_+ + a^2)}{\lambda_+ - \lambda_-} \\
\dfrac{\dot{\lambda}_-}{\lambda_-\partial_- B(\lambda_-)} = -\dfrac{1}{m}\dfrac{4(\lambda_- + a^2)}{\lambda_+ - \lambda_-},
\end{cases} \tag{3.29}
$$

we can get three expressions separated in λ_+, λ_- and φ:

$$\begin{cases} \dfrac{\dot\lambda_+}{\lambda_+\partial_+ A(\lambda_+)} + \dfrac{\dot\lambda_-}{\lambda_-\partial_- B(\lambda_-)} = \dfrac{4}{m} \\ \dfrac{\dot\lambda_+}{\lambda_+(\lambda_+ + a^2)\partial_+ A(\lambda_+)} + \dfrac{\dot\lambda_-}{\lambda_-(\lambda_- + a^2)\partial_- B(\lambda_-)} = 0 \\ -\ell a^2 \left(\dfrac{\dot\lambda_+}{4\lambda_+(\lambda_+ + a^2)^2\partial_+ A(\lambda_+)} + \dfrac{\dot\lambda_-}{4\lambda_-(\lambda_- + a^2)^2\partial_- B(\lambda_-)} \right) = \dot\varphi. \end{cases} \tag{3.30}$$

If the potentials (2.16) are known, e.g., the functions $f(\lambda_+)$ and $g(\lambda_-)$ are chosen, then the functions $\partial_+ A(\lambda_+)$ and $\partial_- B(\lambda_-)$ will be also explicitly known. Thus, integrating over time (3.30) we obtain the following crude expressions:

$$\begin{cases} \displaystyle\int_0^{\lambda_+} \mathrm{d}z \frac{1}{z\partial_z A(z)} + \int_0^{\lambda_-} \mathrm{d}z \frac{1}{z\partial_z B(z)} = \frac{4}{m} t + c_1 \\ \displaystyle\int_0^{\lambda_+} \mathrm{d}z \frac{1}{z(z + a^2)\partial_z A(z)} + \int_0^{\lambda_-} \mathrm{d}z \frac{1}{z(z + a^2)\partial_z B(z)} = c_2 \\ \displaystyle -\ell a^2 \left(\int_0^{\lambda_+} \mathrm{d}z \frac{1}{4z(z + a^2)^2\partial_z A(z)} + \int_0^{\lambda_-} \mathrm{d}z \frac{1}{4z(z + a^2)^2\partial_z B(z)} \right) = \varphi + c_3. \end{cases} \tag{3.31}$$

These relations constitute an implicit form of the integration of the equations of motion. To obtain the coordinates λ_+, λ_- and φ as explicit functions of time we should invert such relations, a task which is not often simple and that will depend on each particular choice of the functions $f(\lambda_+)$ and $g(\lambda_-)$.

3.4. The function F in terms of oblate spheroidal coordinates

From definition (2.9) λ_+ is always positive meanwhile the values of λ_- lie on the interval $[-a^2, 0]$, depending on the scalar product $\mathbf{x} \cdot \mathbf{n}$. Thus, we suggest the following change of coordinates:

$$\lambda_+ = a^2 \sinh^2 \alpha, \qquad \lambda_- = -a^2 \sin^2 \beta. \tag{3.32}$$

Therefore we have

$$x = a \cosh\alpha \sin\beta \cos\varphi, \qquad y = a \cosh\alpha \sin\beta \sin\varphi, \qquad z = a \sinh\alpha \cos\beta, \tag{3.33}$$

where $\{\alpha, \beta, \varphi\}$ are the usual oblate spherical coordinates [5–9]. Here, we shall see how the dependence of F in terms of α, β and φ gives a new insight into the above discussion. If we take time derivative in (3.32), we obtain

$$\dot\lambda_+ = (2a^2 \sinh\alpha \cosh\alpha)\dot\alpha, \qquad \dot\lambda_- = -(2a^2 \sin\beta \cos\beta)\dot\beta. \tag{3.34}$$

From (3.32), one readily obtains

$$4\lambda_+(\lambda_+ + a^2) \left(\frac{\partial F}{\partial \lambda_+} \right)^2 = \left(\frac{\partial F}{\partial \alpha} \right)^2 \tag{3.35}$$

$$4\lambda_-(\lambda_- + a^2) \left(\frac{\partial F}{\partial \lambda_-} \right)^2 = -\left(\frac{\partial F}{\partial \beta} \right)^2. \tag{3.36}$$

Now, we compare the expressions given in (3.27) and (3.32) for $\dot\lambda_+$ and use (3.35) to conclude that

$$\dot\alpha = \frac{1}{m} \frac{1}{\sinh^2\alpha + \sin^2\beta} \frac{\partial F}{\partial \alpha}. \tag{3.37}$$

An analogous manipulation shows that

$$\dot{\beta} = \frac{1}{m} \frac{1}{\sinh^2\alpha + \sin^2\beta} \frac{\partial F}{\partial \beta}. \tag{3.38}$$

Using the last formula in (3.28) and (3.32), we find the expression for the time derivative of φ as

$$\dot{\varphi} = \frac{\ell}{ma^2} \frac{1}{\sinh^2\alpha + \sin^2\beta} \left(\frac{1}{\cos^2\beta} - \frac{1}{\cosh\alpha} \right). \tag{3.39}$$

Since the functions $f(\cdot)$ and $g(\cdot)$ depend respectively on λ_+ and λ_- only, they can be written as functions of α and β, respectively. As $f(\cdot)$ and $g(\cdot)$ are, in principle, arbitrary, we could denote these functions as $f(\alpha)$ and $g(\beta)$, respectively. Thus, (2.15) can be written as (except for an irrelevant change on the sign)

$$\Phi - a^2(\sinh^2\alpha)U = f(\alpha), \qquad \Phi + a^2(\sin^2\beta)U = g(\beta). \tag{3.40}$$

Then, (3.21) with (3.35)–(3.36) and (3.40) gives

$$\left(\frac{\partial F}{\partial \alpha}\right)^2 = 2m[a^2 \sinh^2\alpha E - \widetilde{E} + f(\alpha)] - \ell^2 \frac{\sinh^2\alpha}{\cosh^2\alpha} \tag{3.41}$$

$$\left(\frac{\partial F}{\partial \beta}\right)^2 = 2m[a^2 \sin^2\beta E + \widetilde{E} - g(\beta)] - \ell^2 \frac{\sin^2\beta}{\cos^2\beta}, \tag{3.42}$$

so that, we finally get the following expressions for the derivatives of α and β:

$$\begin{aligned} \dot{\alpha} &= \frac{1}{a^2 m} \frac{1}{\sinh^2\alpha + \sin^2\beta} \left[2m[a^2 \sinh^2\alpha E - \widetilde{E} + f(\alpha)] - \ell^2 \frac{\sinh^2\alpha}{\cosh^2\alpha} \right]^{1/2} \\ &= \frac{1}{a^2 m} \frac{1}{\sinh^2\alpha + \sin^2\beta} \Delta_\alpha^{1/2} \end{aligned} \tag{3.43}$$

$$\begin{aligned} \dot{\beta} &= \frac{1}{a^2 m} \frac{1}{\sinh^2\alpha + \sin^2\beta} \left[2m[a^2 \sin^2\beta E + \widetilde{E} - g(\beta)] - \ell^2 \frac{\sin^2\beta}{\cos^2\beta} \right]^{1/2} \\ &= \frac{1}{a^2 m} \frac{1}{\sinh^2\alpha + \sin^2\beta} \Delta_\beta^{1/2}, \end{aligned} \tag{3.44}$$

where obviously,

$$\Delta_\alpha = 2m[a^2 \sinh^2\alpha E - \widetilde{E} + f(\alpha)] - \ell^2 \frac{\sinh^2\alpha}{\cosh^2\alpha} \tag{3.45}$$

$$\Delta_\beta = 2m[a^2 \sin^2\beta E + \widetilde{E} - g(\beta)] - \ell^2 \frac{\sin^2\beta}{\cos^2\beta}. \tag{3.46}$$

Clearly, after (3.43) and (3.44), we have

$$\frac{\dot{\alpha}}{\Delta_\alpha^{1/2}} - \frac{\dot{\beta}}{\Delta_\beta^{1/2}} = 0 \quad \Longrightarrow \quad \int_0^\alpha \frac{\mathrm{d}\alpha}{\Delta_\alpha^{1/2}} - \int_0^\beta \frac{\mathrm{d}\beta}{\Delta_\beta^{1/2}} = C_2, \tag{3.47}$$

where C_2 is a constant with respect to time (obviously, it depends on α and β). Another constant of motion can be obtained as follows: first, we write (3.39) as

$$\dot{\varphi} = \frac{\ell}{ma^2} \frac{1}{\sinh^2\alpha + \sin^2\beta} \left(\frac{\cos^2\beta + \sin^2\beta}{\cos^2\beta} - \frac{\cosh^2\alpha - \sinh^2\alpha}{\cosh\alpha} \right). \tag{3.48}$$

Then, using (3.43) and (3.44) in (3.48), we have

$$\dot{\varphi} = \frac{\ell}{a^2}\left(\frac{\dot{\beta}}{\Delta_\beta^{1/2}}\frac{\sin^2\beta}{\cos^2\beta} + \frac{\dot{\alpha}}{\Delta_\alpha^{1/2}}\frac{\sinh^2\alpha}{\cosh^2\alpha}\right). \tag{3.49}$$

From (3.49), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\varphi - \frac{\ell}{a^2}\left(\int\frac{\mathrm{d}\alpha}{\Delta_\alpha^{1/2}}\frac{\sinh^2\alpha}{\cosh^2\alpha} + \int\frac{\mathrm{d}\beta}{\Delta_\beta^{1/2}}\frac{\sin^2\beta}{\cos^2\beta}\right)\right] = 0, \tag{3.50}$$

which shows that

$$\varphi - \frac{\ell}{a^2}\left(\int\frac{\mathrm{d}\alpha}{\Delta_\alpha^{1/2}}\frac{\sinh^2\alpha}{\cosh^2\alpha} + \int\frac{\mathrm{d}\beta}{\Delta_\beta^{1/2}}\frac{\sin^2\beta}{\cos^2\beta}\right) = C_3 \tag{3.51}$$

is a constant of motion.

Furthermore, from (3.43) and (3.44) we can obtain a third result as we show in what follows:

$$\frac{\dot{\alpha}}{\Delta_\alpha^{1/2}}\sinh^2\alpha + \frac{\dot{\beta}}{\Delta_\beta^{1/2}}\sin^2\beta = \frac{1}{a^2m}\frac{\sinh^2\alpha}{\sinh^2\alpha + \sin^2\beta} + \frac{1}{a^2m}\frac{\sin^2\beta}{\sinh^2\alpha + \sin^2\beta} = \frac{1}{m}, \tag{3.52}$$

which obviously yields after integration

$$\int\frac{\mathrm{d}\alpha}{\Delta_\alpha^{1/2}}\sinh^2\alpha + \int\frac{\mathrm{d}\beta}{\Delta_\beta^{1/2}}\sin^2\beta = \frac{1}{a^2m}t. \tag{3.53}$$

The function F can now be written in terms of the variables (α, β, φ). Since λ_+ and λ_- are functions of α and β alone, respectively, formula (3.23) can be written as

$$F(\alpha, \beta, \varphi) = A(\alpha) + B(\beta) + \ell\varphi, \tag{3.54}$$

where

$$A(\alpha) = \int\frac{\partial F}{\partial\alpha}\,\mathrm{d}\alpha, \qquad B(\beta) = \int\frac{\partial F}{\partial\beta}\,\mathrm{d}\beta. \tag{3.55}$$

This gives the final expression for $F(\alpha, \beta, \varphi)$ as

$$F(\alpha, \beta, \varphi) = \int\Delta_\alpha^{1/2}\,\mathrm{d}\alpha + \int\Delta_\beta^{1/2}\,\mathrm{d}\beta + \ell\varphi. \tag{3.56}$$

Time invariants C_2 and C_3 can be written in terms of certain partial derivatives of $F(\alpha, \beta, \varphi)$ as we can easily show. In fact, using the expressions for $\Delta_\alpha^{1/2}$ and $\Delta_\beta^{1/2}$ in (3.45) and (3.46), we have that

$$\frac{\partial F}{\partial\widetilde{E}} = -m^2C_2, \qquad \frac{\partial F}{\partial\ell} = C_3, \qquad \frac{\partial F}{\partial E} = t, \tag{3.57}$$

as it can be easily checked. Then, the function F can be written as

$$F \equiv Et - m\widetilde{E}C_2 + \ell C_3, \tag{3.58}$$

where C_2 and C_3 are dependent on α, β and φ, but they are time independent.

4. Quantum systems

From the point of view of quantum mechanics, the Hamiltonian H as well as the integrals of motion L and $\widetilde{H}$ are Hermitian operators obtained simply by replacing $p_k \to -\mathrm{i}\partial_k$, $k = 1, 2, 3$, in (2.1), (2.2) and (2.3), respectively (we have taken $\hbar = 1$ along this section). Formal hermiticity follows from the fact that the operators H, L and $\widetilde{H}$ are symmetric in the usual Cartesian coordinates. The kinetic parts of H and $\widetilde{H}$ are nonsingular quadratic expressions on positions and momenta. This quadratic dependence assures that these kinetic terms are represented by self-adjoint ([11]) operators. In addition, we assume that the potentials $U(\mathbf{x})$ and $\Phi(\mathbf{x})$ satisfy sufficient conditions so that both H and $\widetilde{H}$ be self-adjoint (as for example that the conditions in the Kato– Rellich theorem be satisfied [12]).

The conditions (2.4) on the metric $g(\mathbf{x})$, and (2.6)–(2.7) on the potential terms $U(\mathbf{x})$, $\Phi(\mathbf{x})$ guarantee the commutation relations for these operators:

$$[H, L] = [\widetilde{H}, L] = [H, \widetilde{H}] = 0. \tag{4.1}$$

We will look for the simultaneous eigenfunctions $\psi(\mathbf{x})$ of the three operators

$$(H - E)\psi = (\widetilde{H} - \widetilde{E})\psi = (L - \ell)\psi = 0 \tag{4.2}$$

or equivalently

$$\begin{aligned} -\Delta\psi(\mathbf{x}) &= 2m(E - U(\mathbf{x}))\psi(\mathbf{x}) \\ -\widetilde{\Delta}\psi(\mathbf{x}) &= 2m(\widetilde{E} - \Phi(\mathbf{x}))\psi(\mathbf{x}) \\ -\mathrm{i}\partial_\varphi\psi(\mathbf{x}) &= \ell\psi(\mathbf{x}), \end{aligned} \tag{4.3}$$

where

$$\Delta = \partial_k\partial_k, \qquad \widetilde{\Delta} = \partial_j g^{jk}(\mathbf{x})\partial_k. \tag{4.4}$$

Now, we will express these differential operators in terms of the coordinates λ_+, λ_-, φ in order to rewrite the eigen-equations in the form

$$\begin{aligned} &-\frac{1}{\lambda_+ - \lambda_-}\left\{\left[4\lambda_+(\lambda_+ + a^2)\psi_{++} + 2(a^2 + 3\lambda_+)\psi_+ - \frac{a^2}{\lambda_+ + a^2}\psi_{\varphi\varphi}\right]\right. \\ &\qquad \left. - \left[4\lambda_-(\lambda_- + a^2)\psi_{--} + 2(a^2 + 3\lambda_-)\psi_- - \frac{a^2}{\lambda_- + a^2}\psi_{\varphi\varphi}\right]\right\} = 2m(E - U)\psi \\ &-\frac{\lambda_-}{\lambda_+ - \lambda_-}\left[4\lambda_+(\lambda_+ + a^2)\psi_{++} + 2(a^2 + 3\lambda_+)\psi_+ + \frac{\lambda_+}{\lambda_+ + a^2}\psi_{\varphi\varphi}\right] \\ &\qquad + \frac{\lambda_+}{\lambda_+ - \lambda_-}\left[4\lambda_+(\lambda_+ + a^2)\psi_{--} + 2(a^2 + 3\lambda_-)\psi_- + \frac{\lambda_-}{\lambda_- + a^2}\psi_{\varphi\varphi}\right] = 2m(\widetilde{E} - \Phi)\psi. \end{aligned} \tag{4.5}$$

Taking into account that

$$\psi_{\varphi\varphi} = -\ell^2\psi \tag{4.6}$$

and also (2.14), equations (4.5) can be written in a separated form,

$$\begin{aligned} &-\left[4\lambda_+(\lambda_+ + a^2)\psi_{++} + 2(a^2 + 3\lambda_+)\psi_+ - \frac{\ell^2\lambda_+}{\lambda_+ + a^2}\psi\right] = 2m[(E - U)\lambda_+ - (\widetilde{E} - \Phi)]\psi \\ &-\left[4\lambda_-(\lambda_- + a^2)\psi_{--} + 2(a^2 + 3\lambda_-)\psi_- - \frac{\ell^2\lambda_-}{\lambda_- + a^2}\psi\right] = 2m[(E - U)\lambda_- - (\widetilde{E} - \Phi)]\psi, \end{aligned} \tag{4.7}$$

where λ_+ and λ_- appear in the first and second equations in (4.7), respectively. Therefore we can look for a factorized solution for the eigenfunctions as follows:

$$\psi(\mathbf{x}) = \psi^+(\lambda_+)\psi^-(\lambda_-)\,\mathrm{e}^{\mathrm{i}\ell\varphi}. \tag{4.8}$$

We can go back to the change of coordinates given by (3.32). This change of coordinates is also suggested by the formulae below, as we shall see. Now, it is time for choosing explicit forms for the functions $f(x)$ and $g(x)$ in (2.15). For $f(x)$, we shall choose the function that vanishes identically. For $g(x)$, we choose $g(\lambda_-) := -Q(\lambda_- + a^2)$, where Q is a constant. Then, the following expressions arise:

$$\begin{array}{ll} f(\lambda_+) = \Phi - \lambda_+ U = 0, & g(\lambda_-) = \Phi - \lambda_- U = -Q(\lambda_- + a^2) \\ U(\mathbf{x}) = -Q\dfrac{\lambda_- + a^2}{\lambda_+ - \lambda_-}, & \Phi(\mathbf{x}) = -Q\dfrac{\lambda_+(\lambda_- + a^2)}{\lambda_+ - \lambda_-} \end{array}. \tag{4.9}$$

Or, in Cartesian coordinates,

$$U(\mathbf{x}) = -\frac{Q}{2}\left(\frac{\mathbf{x}^2 + a^2}{[(\mathbf{x}^2 + a^2)^2 - 4a^2(\mathbf{x}^2 - (\mathbf{x}\cdot\mathbf{n})^2)]^{1/2}} - 1\right). \tag{4.10}$$

Now, we carry (3.32) and (4.9) into (4.7) to get the following set of two differential equations in the variables α and β:

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2} + \frac{\sinh\alpha}{\cosh\alpha}\frac{\mathrm{d}}{\mathrm{d}\alpha} - \frac{\sinh^2\alpha}{\cosh^2\alpha}\ell^2 + 2m[Ea^2\sinh^2\alpha - \widetilde{E}]\right\}\psi^+(\alpha) = 0, \tag{4.11}$$

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}\beta^2} - \frac{\sin\beta}{\cos\beta}\frac{\mathrm{d}}{\mathrm{d}\beta} - \frac{\sin^2\beta}{\cos^2\beta}\ell^2 + 2m[Ea^2\sin^2\beta + \widetilde{E} + Qa^2\cos^2\beta]\right\}\psi^-(\beta) = 0. \tag{4.12}$$

This is a special type of equations already studied in the literature that we briefly analyze in the next section. In figure 1 we depict three possible situations: $f = 1, g = 0$; $f = 0, g = 1$ and $f = 1, g = 1$ respectively from left to right and from above to below.

5. Study of the equations and their solutions

First of all, it seems convenient to simplify equations (4.11)–(4.12). In order to fulfil this goal, let us choose the following new coordinates:

$$t = \sinh\alpha, \qquad u = \sin\beta, \qquad \frac{\mathrm{d}}{\mathrm{d}\alpha} = \sqrt{t^2+1}\,\frac{\mathrm{d}}{\mathrm{d}t}, \qquad \frac{\mathrm{d}}{\mathrm{d}\beta} = \sqrt{1-u^2}\,\frac{\mathrm{d}}{\mathrm{d}u}. \tag{5.1}$$

Let us introduce the following parameters:

$$\mathcal{E} = 2ma^2E, \qquad q = 2ma^2Q, \qquad \widetilde{\mathcal{E}} = \ell^2 + 2m\widetilde{E}, \tag{5.2}$$

then, equations (4.11)–(4.12) become respectively

$$\left[(t^2+1)\frac{\mathrm{d}^2}{\mathrm{d}t^2} + 2t\frac{\mathrm{d}}{\mathrm{d}t} + \frac{\ell^2}{t^2+1} + \mathcal{E}t^2 - \widetilde{\mathcal{E}}\right]\psi^+(t) = 0. \tag{5.3}$$

$$\left[(1-u^2)\frac{\mathrm{d}^2}{\mathrm{d}u^2} - 2u\frac{\mathrm{d}}{\mathrm{d}u} - \frac{\ell^2}{1-u^2} + \mathcal{E}u^2 + q(1-u^2) + \widetilde{\mathcal{E}}\right]\psi^-(u) = 0. \tag{5.4}$$

We observe that in the case of $\mathcal{E} = q = 0$ and only in this case, these equations can be reduced to equations of hypergeometric type, which can be solved in terms of hypergeometric

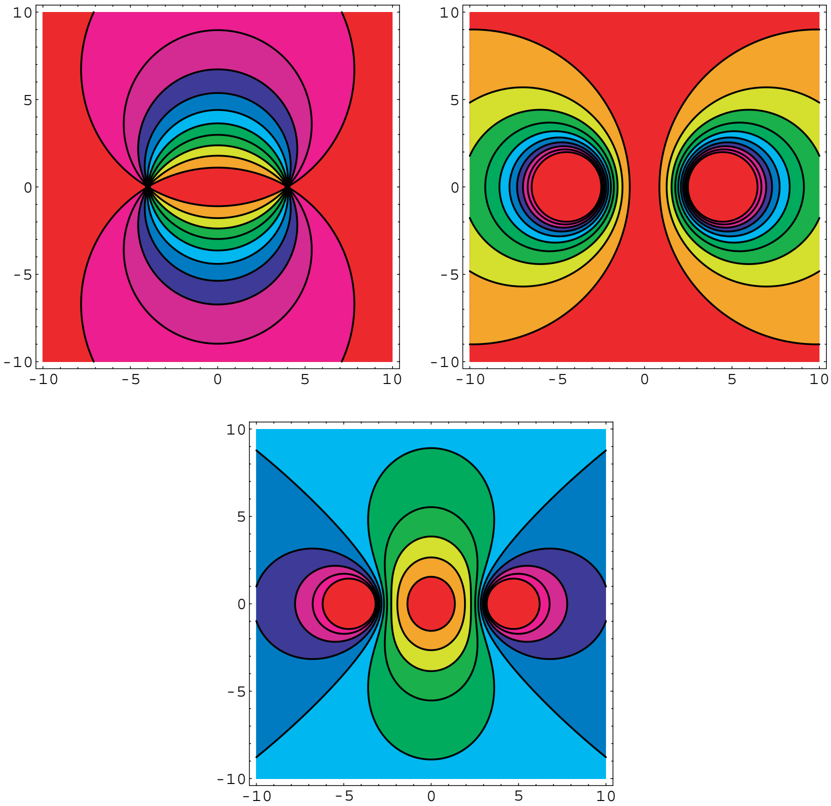

Figure 1. Iso-potential lines in the XZ-plane for three values of the parameters.
(This figure is in colour only in the electronic version)

functions. However, this is not the most general case, let us consider the following differential equation:

$$(1-z^2)\frac{d^2 w}{dz^2} - 2z\frac{dw}{dz} + \left\{\lambda - \frac{\mu^2}{1-z^2} + \gamma^2(1-z^2)\right\} w = 0, \tag{5.5}$$

where λ, μ and γ^2 are real parameters (γ^2 may be positive or negative depending on the use of prolate or oblate coordinates respectively). This is called the *spheroidal wave equation* [10]. It is very simple to show that both equations (5.3) and (5.4) are versions of the spheroidal wave equation. In fact, (5.4) can be written as

$$(1-u^2)\frac{d^2\psi^-(u)}{du^2} - 2u\frac{d\psi^-(u)}{du} + \left\{G - \frac{\ell^2}{1-u^2} + q'(1-u^2)\right\}\psi^-(u) = 0, \tag{5.6}$$

with $G := \widetilde{\mathcal{E}} + \mathcal{E}$ and $q' = q - \mathcal{E}$. If we apply the change of variables given by $t = i\alpha$, equation (5.3) becomes

$$(1-\alpha^2)\frac{d^2\psi^+(\alpha)}{d\alpha^2} - 2\alpha\frac{d\psi^+(\alpha)}{d\alpha} + \left\{G - \frac{\ell^2}{1-\alpha^2} - \mathcal{E}(1-\alpha^2)\right\}\psi^+(\alpha), \tag{5.7}$$

where again $G = \widetilde{\mathcal{E}} + \mathcal{E}$ and we have kept the notation $\psi^+(\alpha) = \psi^+(i\alpha) = \psi^+(t)$.

Solutions of the spheroidal wave equation (5.5) and therefore of (5.6) and (5.7) have been studied in [10]. The origin $z = 0$ is a regular point of the equation and therefore, we can find two linearly independent functions in terms of power series on the variable z. These series converge on the open circle centered at the origin and radius equal to one, since $z = \pm 1$ are singular points for the equation. On this open circle, one can find one even and one odd solution of (5.5) of the form $\sum_{n=0}^{\infty} a_n z^{2n}$ and $\sum_{n=0}^{\infty} a_n z^{2n+1}$ respectively, which are linearly independent. These series do not converge at the singular points ± 1. As they do converge on the open interval $(-1, 1)$, the wavefunction $\psi^{-}(\beta)$ solution of equation (4.12) is periodic on the real axis with singularities at the points $(2n+1)\pi/2$. There exists another type of linearly independent even and odd solutions on the neighborhood of the origin that may converge at the singular points ± 1, provided that a relation is satisfied between the coefficients λ, μ and γ in (5.5) (and its corresponding translation in terms of the coefficients in (5.6) and (5.7)) [10]. In any case, the solutions $\psi^{-}(\beta)$ of (4.12) on the real axis are periodic and therefore, not square integrable.

With respect to equation (5.5), $z = \pm 1$ are regular singular points with indices equal to $\pm\frac{1}{2}\mu = \pm\frac{1}{2}\ell$. Being ℓ an integer, the two linearly independent solutions on the neighborhood of $z = 1$ are $u_1(z)$ and $u_2(z)$ with $u_1(z) = (z-1)^{\ell/2} F_1(z-1)$ and $u_2(z) = u_1(z)\log(z-1) + (z-1)^{-\ell/2} F_2(z-1)$, where $F_1(z-1)$ and $F_2(z-1)$ are power series on $z-1$. These power series have radii of convergence equal to 2. On the neighborhood of $z = -1$, similar solutions can be found. Power series never truncate.

There is another singular point at $z = \infty$. This singular point is irregular. Solutions on the neighborhood of the infinite have the form $z^{\nu} \sum_{n=-\infty}^{\infty} a_n z^{2n}$, where ν is a complex number depending on the equation parameters. In order to simplify the recurrence relations for the coefficients a_n, it is customary to choose this solution as $(z^2-1)^{\mu/2} z^{\nu-\mu} \sum_{n=-\infty}^{\infty} a_n z^{2n}$. The condition that the Laurent series converges in $1 < |z| < \infty$ gives a relation between λ, μ, γ and ν [10]. These solutions are of the form

$$(z^2-1)^{\mu/2} z^{\mu} \sum_{n=-\infty}^{\infty} a_{\nu,n}^{\mu} \psi_{\nu+2n}^{(j)}(\gamma z), \qquad j = 1, 2, 3, 4, \tag{5.8}$$

where

$$\psi_{\nu}^{(j)}(z) = \left(\frac{\pi}{2z}\right)^{1/2} Z_{\nu}^{(j)}(z), \tag{5.9}$$

with $Z_{\nu}^{(1)}(z) = J_{\nu}(z)$, $Z_{\nu}^{(2)}(z) = Y_{\nu}(z)$, $Z_{\nu}^{(3)}(z) = H_{\nu}^{(1)}(z)$ and $Z_{\nu}^{(4)}(z) = H_{\nu}^{(2)}(z)$, being $J_{\nu}(z)$, $Y_{\nu}(z)$ and $H_{\nu}^{(i)}(z)$ the Bessel functions of first, second and third class, respectively. Any two of the set of solutions (5.8) are linearly independent provided that ν be not a half odd integer.

Solutions of (5.3) and (5.4) can be obtained without resorting to the standard study of the spheroidal wavefunction. For instance, if we use the change of variables given by $z := t^2$ in (5.3), this equation is transformed into

$$\left[4z(z+1)\frac{\mathrm{d}^2}{\mathrm{d}z^2} + (6z+2)\frac{\mathrm{d}}{\mathrm{d}z} + \frac{\ell^2}{z+1} + \mathcal{E}z - \widetilde{\mathcal{E}}\right]\psi(z) = 0. \tag{5.10}$$

Now, the singular regular points lie at $z = 0, -1$ and it is not difficult to obtain solutions in the form of power series on the neighborhood of these points. For example, for $z = 0$ the characteristic exponents are 0 and $1/2$ giving respective linearly independent solutions of (5.10) of the form $\psi_0(z) = \sum_{n=0}^{\infty} a_n z^n$ and $\psi_{1/2}(z) = \sum_{n=0}^{\infty} b_n z^{n+\frac{1}{2}}$. Recurrence relations for the coefficients depend on four coefficients, except the first and second relations which depend on the two and three first coefficients respectively (which is compatible with the fact that a_0 and b_0 should be the only independent coefficients). On the neighborhood of the singular point $z = -1$, two linearly independent solutions can be found of the form $\psi_1(z) = \sum_{n=0}^{\infty} a_n (z+1)^{n+\ell/2}$ and $\psi_2(z) = \psi_1(z)\log(z+1) + \sum_{n=0}^{\infty} b_n (z+1)^{n-\ell/2}$. These series make sense provided that compatibility relations exist between the parameters ℓ, $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ in complete agreement with the general study of the solutions of spheroidal wavefunctions in [10].

6. Conclusions and remarks

We have studied the conditions of integrability of a classical or quantum system having a symmetry axis. As in a three-dimensional integrable system, we have found three independent observables such that their respective Poisson brackets are zero, in the classical case, or commute in the quantum case. The chosen symmetry forces one of the observables to be the component of the angular momentum in the direction of the symmetry axis. The other two can be written in the Hamiltonian form as a sum of a kinetic term plus a potential.

In the classical case, we have obtained the most general form of the potentials corresponding to both Hamiltonians in terms of oblate spheroidal coordinates, that depend on two arbitrary functions depending on one coordinate only. We have written the equations of motion in terms of these coordinates and show that the Hamilton–Jacobi characteristic function can be written as a sum of three functions each one depending on one coordinate only. Then, we have obtained the explicit form for these three functions.

The quantum case is obtained by direct canonical quantization of the classical case. The condition of integrability yields two Schrödiger-type equations with separate variables. Then, a reasonable choice on the functions that determine the potentials yields new equations that are shown to be of the spheroidal type. We finish the discussion with some comments on the solutions of this kind of equations.

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